Stratified Generalized Procrustes Analysis

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In many different problems, data analysis requires one to first compensate for a global transformation between the different datasets of *shape data*. This is known as *procrustes analysis* in the statistics and shape analysis literature [1, 3]. More precisely, it is called *generalized procrustes analysis* when more than two shape data are to be registered. In this problem, one global transformation per observed shape has to be computed, so that the shapes are mapped to a common coordinate frame whereby they look as 'similar' as possible. This process is called also *rigid registration*.

The classical approach to generalized procrustes analysis is to select one of the shapes as a *reference shape*, and register each of the other shapes to the reference in turn by solving the absolute orientation problem. It is common to then alternate a re-estimation of the reference shape, as the average of the registered shapes, with shape registration. We call this general paradigm the *alternation approach* to generalized procrustes analysis. Both iterative [2] and algebraic closed-form solutions [4] were proposed for the absolute orientation problem. Integrated solutions based on the alternation approach for multiple shapes have been recently proposed in [5, 6] with a total least squares step.

We propose a *stratified approach* inspired by recent results obtained in Structure-from-Motion. Our stratified approach offers a statistically grounded framework for obtaining both the transformations and the reference shape at once in two steps. First, we compute a reference shape and affine transformations. Second, we upgrade these transformations to the sought after similarity or euclidean transformations. In practice each of these two steps involves solving a non-convex optimization problem. We provide convex approximations and closed-form solutions.

As opposed to the classical alternation approach, our stratified approach processes data in batch. It gracefully deals with missing data.

We assume that there exist an unknown reference shape $\mathsf{S}^\top \in \mathbb{R}^{d\times m}$, $\mathbf{S}_j \in \mathbb{R}^d$, and unknown global transformations $\mathscr{T} \stackrel{\text{def}}{=} \{T_1,\ldots,T_n\}$, $T_i : \mathbb{R}^d \to \mathbb{R}^d$, such that the discrepancy between the model predicted shape points $T_i(\mathbf{S}_j) \in \mathbb{R}^d$ and the observed shape points $\mathbf{D}_{i,j} \in \mathbb{R}^d$ follows a gaussian *i.i.d* distribution of unknown variance σ^2. Maximizing the likelihood of the difference between the shapes D_i and the transformed reference shape $T_i(\mathbf{S}_j)$ amounts to minimizing the negative log likelihood, proportional to the *data-space cost* $\mathscr{E}$, defined by:

$$\mathscr{E}(\mathscr{T},\mathsf{S}) \stackrel{\text{def}}{=} \sum_{i=1}^{n}\sum_{j=1}^{m} v_{i,j}\, \|\mathbf{D}_{i,j} - T_i(\mathbf{S}_j)\|_2^2. \qquad (1)$$

The variables $v_{i,j} \in \{0,1\}$ allow us to model *missing data*. The data-space cost is related to a generative modeling of the data, that we hereinafter call the *data-space model*. It matches the definition of [7] of registration based on the notion of average shape and it is gauge invariant (See figure 1 for a graphical representation).

Our framework has four main steps. The first two steps perform affine registration (initialization and nonlinear refinement, respectively), and are needed in all cases, except registration without missing data that is dealt with matrix factorization. The last two steps perform similarity-euclidean registration (initialization from the affine registration and nonlinear refinement, respectively).

We define $\mathscr{A} \stackrel{\text{def}}{=} \{A_1,\ldots,A_n\}$ to be the set of unknown affine transformations. Substituting the affine transformation model into the general data-space cost (1) gives:

$$\mathscr{E}(\mathscr{A},\mathsf{S}) = \sum_{i=1}^{n}\sum_{j=1}^{m} v_{i,j}\, \|\mathbf{D}_{i,j} - \mathsf{A}_i\mathbf{S}_j - \mathbf{a}_i\|_2^2. \qquad (2)$$

Due to the non-convex nature of the *data-space* cost, for the general case our strategy is to get an initial (approximate) closed-form solution based on the following *reference-space model*. The data-space cost $\mathscr{E}$ by what we call the reference-space cost $\tilde{\mathscr{E}}$:

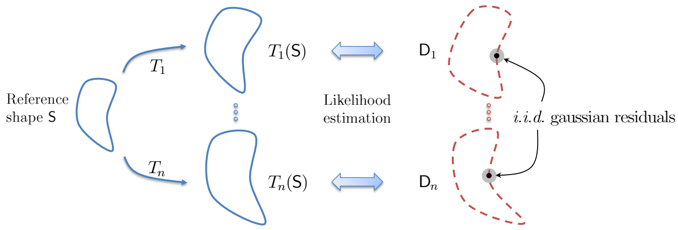

Figure 1: The data-space model.

$$\mathscr{E}(\mathscr{A},\mathsf{S}) \approx \tilde{\mathscr{E}}(\mathscr{B},\mathsf{S}) \stackrel{\text{def}}{=} \sum_{i=1}^{n}\sum_{j=1}^{m} v_{i,j}\, \|\mathsf{B}_i\mathbf{D}_{i,j} + \mathbf{b}_i - \mathbf{S}_j\|_2^2. \qquad (3)$$

where we use the inverse $B_i = (\mathsf{B}_i;\mathbf{b}_i) \stackrel{\text{def}}{=} (\mathsf{A}_i^{-1}; -\mathsf{A}_i^{-1}\mathbf{a}_i)$ of the sought affine transformations. The major advantage of the reference-space cost is that it is a sum of squares linear in the adjustable parameters, and thus leads to a linear least squares optimization problem. Given the initial suboptimal but closed-form solution in the reference-space, we propose to make iterative refinements through efficient nonlinear least squares algorithms such as Gauss-Newton, Levenberg-Marquardt and other variants

After computing the set of affine transformations $A_i = (\mathsf{A}_i;\mathbf{a}_i)$, $i = 1,\ldots,n$ a set of similarity/euclidean transformations are computed. In the data-space model framework, the affine registration we compute is up to an unknown global affine transformation $G = (\mathsf{G};\mathbf{g})$. The rotational part of this affine transformation factors in the product of a euclidean Q and a purely affine transformation Z: $\mathsf{G} = \mathsf{QZ}$. The upgrading transformation is represented by Z. The projection to Stiefel manifold must take place after upgrading. After the upgrading the iterative refinement is similar to the affine case.

Overall, our methods are very efficient and resistant to various pertubation (extreme noise and amounts of missing data). We provide results on synthetic and real data sets. Compared to the alternation schema, our algorithm obtains lower error in both affine and euclidean cases, especially for shapes with high deformations.

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