# An implicit non-linear numerical scheme for illumination-robust variational optical flow


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Optical flow describes the apparent motion observed in a sequence of images. We present a novel numerical scheme for variational optical flow calculation that makes no assumption on the analytical form of the employed optical flow constraint. The scheme works with different smoothness criteria combining local and global methods in a natural way. We use this framework to formulate an illumination-robust optical flow calculation method based on normalised mean-shifted cross-correlation.

The variational problem for optical flow calculation is often formulated as finding the displacement function $\mathbf{u}(\mathbf{x})$ that minimises a functional of the form

$$\mathscr{F}(\mathbf{u}) = \int_{\mathbf{x}} \left( E(\mathbf{x},\mathbf{u}) + \frac{\lambda}{2} \left( |\nabla_{\mathbf{x}} u|^2 + |\nabla_{\mathbf{x}} v|^2 \right) \right), \qquad (1)$$

where $E(\mathbf{x},\mathbf{u})$, $\mathbf{x} = (x,y)$, $\mathbf{u} = (u,v)$, is a function describing optical constraints, while $\lambda$ is a parameter.

Most of the methods use an iterative solver that improves the optical flow estimate obtained in a previous step as $\mathbf{u} \to \mathbf{u}'$, repeating the procedure until a steady state is reached. For finding $\mathbf{u}'$, at each image point $\mathbf{x}$ we have to solve the two-dimensional root-finding problem

$$\mathbf{g}' - \lambda\left(\bar{\mathbf{u}} - 4\mathbf{u}'\right) = 0, \qquad (2)$$

where $\mathbf{g}'$ is $\nabla_{\mathbf{u}} E$ computed in $\mathbf{u}'$ and $\bar{\mathbf{u}}(x,y) = \mathbf{u}(x-1,y) + \mathbf{u}(x,y-1) + \mathbf{u}(x+1,y) + \mathbf{u}(x,y+1)$. $\mathbf{g}'$ is not necessarily linear, but one can choose $E$ such that it becomes linear. Horn and Schunck [1] proposed

$$E_{\mathrm{HS}} = \frac{1}{2}\left(I_t + uI_x + vI_y\right)^2 = \frac{1}{2}\left(I_t + \mathbf{u}\nabla_{\mathbf{x}} I\right)^2, \qquad (3)$$

where $I(\mathbf{x},t)$ is the image brightness. Minimising $E_{\mathrm{HS}}$ approximates the brightness constancy assumption. Solving Eq. (2) then results in

$$\mathbf{u}' = \mathbf{A}^{-1}\left(\lambda\bar{\mathbf{u}} - \tilde{\mathbf{g}}\right), \qquad (4)$$

where $\tilde{\mathbf{g}} = I_t \nabla_{\mathbf{x}} I$ and $\mathbf{A} = \mathbf{H} + 4\lambda\mathbf{I}$. Here the matrix $\mathbf{H} = H_{\alpha\beta} = I_\alpha I_\beta$, $\alpha, \beta \in \{x,y\}$, is formed by brightness derivatives, while $\mathbf{I}$ is a unit matrix.

Our scheme is similar to the Horn-Schunck scheme, however, the root-finding problem formulated in Eq. (2) is solved with Newton's method, while $E$ is calculated only for integer-valued velocities and bicubic interpolation is used to find its first- and second-order derivatives.

For this, we consider $u^{(0,0)} = \lfloor u \rfloor$, $v^{(0,0)} = \lfloor v \rfloor$ and its integer-valued neighbourhood as illustrated in Fig. 1. At given $\mathbf{x}$, we calculate the values of $E$ for $\mathbf{u}^{(i,j)}$, $E_{\mathbf{u}}^{(i,j)} = E\left(\mathbf{x}, \mathbf{u}^{(i,j)}\right)$, as well as the approximate values of the first-order derivatives $E_u$, $E_v$ and the cross-derivative $E_{uv}$.

After computing the derivatives for all $i,j \in \{0,1\}$, we can calculate the bicubic interpolation that approximates $E$ and estimate the first- and second-order derivatives of $E$ at any $\mathbf{u}$ in the cell delimited by the velocity points $(0,0)$, $(0,1)$, $(1,0)$, and $(1,1)$. (See Fig. 1.)

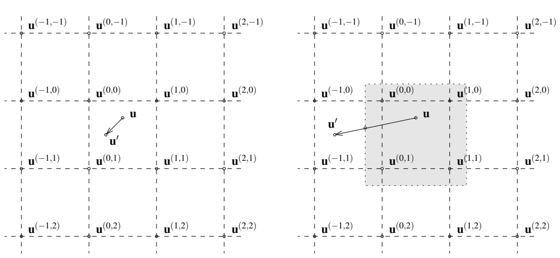
As the bicubic interpolation maintains the continuity of first-order derivatives across cell boundaries, we can solve the non-linear root finding problem given in Eq. (2) with Newton's method. Introducing the notations $\mathbf{g} = \nabla_{\mathbf{u}} E$ and $\mathbf{H} = E_{\upsilon\nu}$, where $\upsilon, \nu \in \{u,v\}$, we obtain the fixed-point equation

$$\mathbf{u}' = \mathbf{A}^{-1}\left(\lambda\bar{\mathbf{u}} - \mathbf{g} + \mathbf{H}\mathbf{u}\right), \qquad (5)$$

where $\mathbf{A} = \mathbf{H} + 4\lambda\mathbf{I}$.

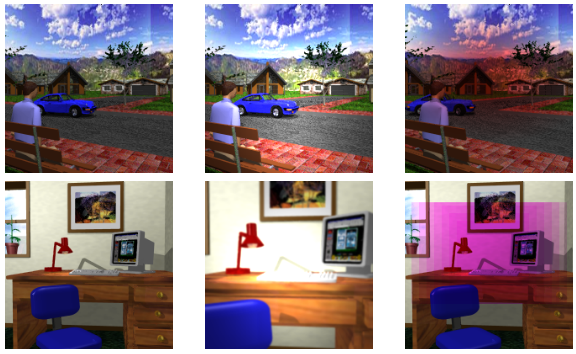
If the gradients are too steep, typically for regions where intensity changes cannot be modelled with optical flow, numerical instability may develop. To overcome this, we introduce a step tolerance and clip $\mathbf{u}'$ to the corresponding tolerance region.

The proposed scheme can be extended to multiple image components $I^m$ such as $RGB$, the gradients $I_x, I_y$, the Laplacian $\Delta I$, or the spherical colour coordinates $\rho$, $\theta$, $\phi$ [2]. In our experimental study, we compare different components and two metrics, cross-correlation and $L_1$, on standard test images with artificial brightness and colour illumination changes (effects) introduced. It is shown that using cross-correlation in a small window improves robustness to both kinds of illumination changes.



integer-valued velocity grid | step tolerance

Figure 1: Integer-valued velocity grid and step tolerance.



original | brightness | colour

Figure 2: STREET and OFFICE test images with illumination changes.

STREET

| effect | CC | $L_1$ |
|---|---|---|
| none | 5.44° | 5.38° |
| bright. | 5.48° | 14.38° |
| colour | 5.43° | 7.63° |

OFFICE

| effect | CC | $L_1$ |
|---|---|---|
| none | 5.56° | 7.22° |
| bright. | 5.91° | 17.45° |
| colour | 5.64° | 11.95° |

Table 1: Average angular errors for $R, G, B$.

STREET

| effect | CC | $L_1$ |
|---|---|---|
| none | 5.37° | 5.45° |
| bright. | 5.37° | 5.92° |
| colour | 5.35° | 7.84° |

OFFICE

| effect | CC | $L_1$ |
|---|---|---|
| none | 5.53° | 8.16° |
| bright. | 5.90° | 8.45° |
| colour | 6.39° | 10.93° |

Table 2: Average angular errors for $\rho, \theta, \phi$.

STREET

| effect | CC | $L_1$ |
|---|---|---|
| none | 4.81° | 5.25° |
| bright. | 4.82° | 5.97° |
| colour | 4.82° | 5.19° |

OFFICE

| effect | CC | $L_1$ |
|---|---|---|
| none | 5.02° | 7.09° |
| bright. | 5.00° | 8.80° |
| colour | 5.21° | 6.92° |

Table 3: Average angular errors for $R, G, B, I_x, I_y$.


[1] B. K. P. Horn and B. G. Schunck. Determining optical flow. *Artificial Intelligence*, 17:185–203, 1981.

[2] Y. Mileva, A. Bruhn, and J. Weickert. Illumination-robust variational optical flow with photometric invariants. In *DAGM-Symposium*, pages 152–162, 2007.