Phase from Zero-Crossings

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The theory of zero-crossing detection is extended to include the first and second spatial derivative of the gaussian distribution. On the premise that phase contains a considerable proportion of the information content in an unknown signal, we show that phase can be extracted from the first and second derivatives of an appropriate filter. It is shown that the spatial gradient of phase can be used to obtain an estimate for the local spectral properties of a signal. By assigning an upper and lower frequency cut-off to each filter, it is suggested that false zero-crossings can be removed from analysis.

1 INTRODUCTION

The computational theory of early visual processing developed by Marr and his associates [4], suggested that the zero-crossings obtained by convolving the $\nabla^2 G$ operator with an image function forms the basis for early visual edge detection. To extend Marr's approach, we recall the early notions of Gabor [1], who proposed representing a real signal of the form :

$$s(x) = a\cos(\omega x) + b\sin(\omega x) \qquad (1)$$

by the complex form:

$$\Phi(x) = s(x) + j\sigma(x) = (a - jb)\exp(j\omega x) \qquad (2)$$

In his words, The function $\sigma(x)$ is formed from $s(x)$ which "*represents the signal $s(x)$ in quadrature, which when added to it , transforms the oscillating vector into a rotating vector* " . We combine the work of Marr and Gabor. To generalise, consider a real even function, such that the Fourier transform of the first and second spatial derivatives exist and are integrable in the usual sense and:

$$\sqrt{x}f(x) \rightarrow 0 \quad as \quad |x| \rightarrow \infty$$

By the Fourier derivative theorem, we have:

$$f^{(1)}(x) \Rightarrow j\omega F(\omega) \qquad (3)$$

and

$$f^{(2)}(x) \Rightarrow -\omega^2 F(\omega) \qquad (4)$$

*This research was supported by a joint SERC Fellowship to the Dept. of Engineering Science and Exp. Psychology at the the University of Oxford.

AVC 1989 doi:10.5244/C.3.50

If we consider convolving the above equations in the spatial domain with a single cosine grating then we obtain as an equivalent representation in Frequency space:

$$\mathcal{F}(f^{(1)}(x) * \cos(\omega_o x)) = j\frac{1}{2}\omega F(\omega) \quad (5)$$

$$(\delta(\omega + \omega_o) + \delta(\omega - \omega_o))$$

and

$$\mathcal{F}(f^{(2)}(x) * \cos(\omega_o x)) = -\frac{1}{2}\omega^2 F(\omega) \quad (6)$$

$$(\delta(\omega + \omega_o) + \delta(\omega - \omega_o))$$

The closed form solution to these equations can now be found in the signal space. The simplest route is via the IFT of the above equation. If we let I_a and I_s, represent the solutions to the convolution of the first and second derivatives respectively, then we find that:

$$I_s = \frac{1}{2}\int_{-\infty}^{\infty} -\omega^2 F(\omega)(\delta(\omega+\omega_o)+\delta(\omega-\omega_o)) \quad (7)$$

$$\exp[j\omega x] \; d\omega$$

$$I_a = \frac{1}{2}\int_{-\infty}^{\infty} j\omega F(\omega)(\delta(\omega+\omega_o)+\delta(\omega-\omega_o)) \quad (8)$$

$$\exp[j\omega x] \; d\omega$$

Using the property that $F(\omega) = F(-\omega)$ because $f(x)$ is real and even, then:

$$\Theta(x) = \frac{I_a}{I_s} = \frac{-1}{\omega_o} \tan(\omega_o x) \quad (9)$$

The first and second derivatives of an appropriate function can therefore be represented in phase space. The importance of phase in signal processing has already been applied to the stereoscopic correspondence problem [3]. It would also be appropriate to quote Lange [2] , who wrote " *if it is possible to allot a mean frequency, with defined phase position to a narrow spectrum of a fluctuating process, then it is also possible to allocate a certain reciprocal phase position to two fluctuating processes of the same frequency band.* ". In addition, we can easily show from the disparity gradient limit, that signals differing in spatial frequency by more than 1.25 octaves violate a disparity gradient limit of 1.

Having obtained a defined phase position, we will now show a simple method for spectral analysis. Keeping the same notation, it is easy to show that:

$$\frac{d[\tan^{-1}(\Theta(x))]}{dx} = \frac{\omega_o}{\frac{1}{\zeta}\sin^2(\omega_o x) + \zeta\cos^2(\omega_o x)} \quad (10)$$

Where ω_o is the instantaneous frequency of the signal under analysis, and ζ represents a skewed non-linearity in phase, which in this case is also equal to the frequency under analysis. Thus a filter operating under the conditions of non-linear phase response, introduces additional oscillatory behaviour. From this type of response, it would be difficult to isolate the spectral component alone. However, by utilising the zero-crossing elements from the real and imaginary components, we notice that the non-linear term is of no consequence under these conditions. This is because the zero-crossings described, correspond to the poles and zeros of equation 9. Therefore, referring to equation 9 in phase space, the zeros represent 0 and π rads, and the poles represent +/- $\frac{\pi}{2}$ rads from the convolutions with the first and second derivatives respectively. Thus it is possible to interpolate between zero-crossing elements of the real and imaginary components and obtain the spectral component from the gradient, which is equivalent to interpreting the displacement of zero-crossings as a spatial wavelength.

We notice in passing, that the same principle could be applied in reverse towards the maximal energy response from each separate element, which we would expect to occur in quadrature with Zero-crossings. This is, however, not always the case and a current area of investigation. We should also observe that signed responses from these filter pairs represented as an arc tangent may be either +/- , +/+ , -/+ or -/- which only occurs in a unique region of the filter pairs receptive field. There is therefore no phase wrap around attributed to the filter alone. This is not the case for oscillatory based quadrature filter pairs unless they are truncated.

By assigning an upper and lower frequency cut-off based upon the bandwidth of the current filter pair, we can effectively filter out regions of the image which do not satisfy the expected frequency range for each band-pass filter. This is of no concern, since we would apply several band-pass filters in general.

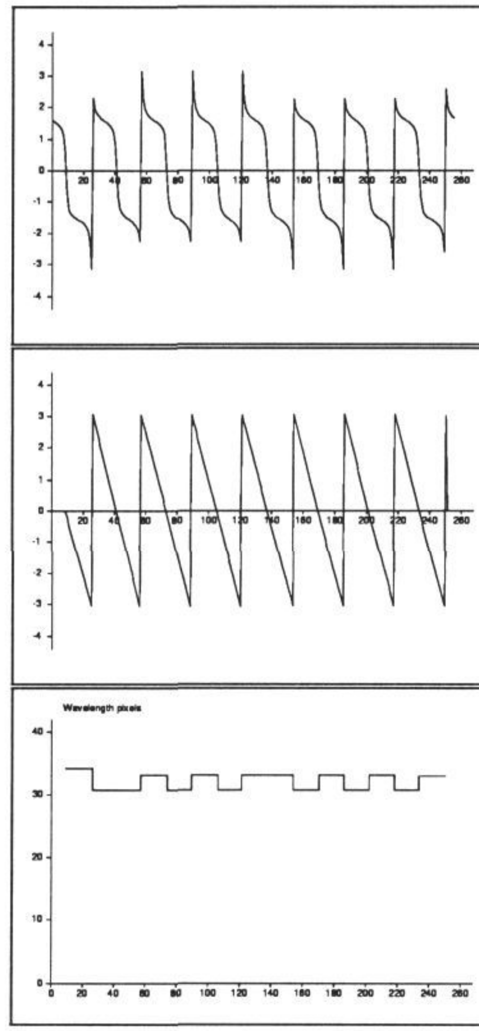

Figure 1: *(a) Reconstruction of phase from a sinusoidal signal of 1/32 cpp (Cycles per Pixel). (b) Linear phase reconstructed from Zero-crossings.(c) Spectral estimates from Zero-crossings shown as a wavelength for clarity. Error in measurement found to be +/- 3.1%.*

2 RESULTS

Results are presented for the phasic response to a sinusoidal grating. Figure 1.a shows the phasic response without the knowledge of the spectral properties of the sinusoidal signal. In figure 1.b we present the reconstruction of linear phase of an unknown sinusoidal grating from zero-crossings and the spectral estimates (fig1.c). Figure 2 shows the the application of this technique to real data. Notice that we have used the upper and lower cut-off frequencies of the filter to remove regions where we might expect false zero-crossings to occur, and isolated image regions with the spatial frequencies of interest.

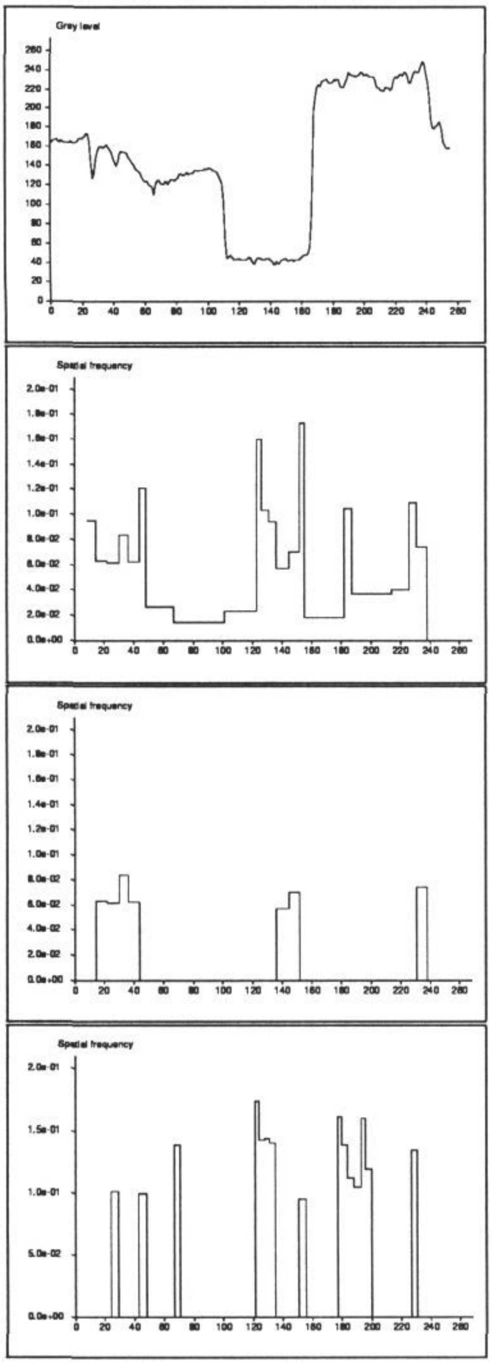

Figure 2: *(a) Raster line scan of a real image. (b)Spectral estimates from Zero-crossings for filter pairs weighted towards 1/16 cpp. (c) Thresholding based upon a lower and upper cut-off of 1/2 octave from fig4.b. (d) Thresholding applied for filter pairs weigthed towards 1/8 cpp.*

3 CONCLUSION

That zero-crossing elements can be used from first and second derivatives to reconstruct phase space is unexpected. However, the phase response is highly nonlinear from these filter pairs. We have shown that it is possible to achieve an approximation to linear phase using the properties of zero-crossings from the 1st and second spatial derivatives of the Gaussian distribution. More simply, the results imply that the spatial separation of edges, as opposed to edges themselves may well provide a useful matching criterion for correspondence problems.

References

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